COBEM 99
XV Congresso Brasileiro de Engenharia Mecânica
15th Brazilian Congress of Mechanical Engineering
22 - 26 de Novembro de 1999 / November 22 - 26, 1999 Águas de Lindóia, São Paulo,

NUMERICAL INVESTIGATION OF NONLINEAR EVOLUTION OF THREE-DIMENSIONAL WAVETRAINS IN FLAT PLATE BOUNDARY LAYERS

Márcio T. Mendonça
Centro Técnico Aeroespacial - Instituto de Atividades Espaciais
Pça Mal. Eduardo Gomes, 50 - 12228-904 - São José dos Campos - SP - Brazil
email: marcio_tm@yahoo.com
Marcello A. Faraco de Medeiros
Pontifícia Universidade Católica de Minas Gerais - Departamento de Engenharia Mecânica
Av. Dom José Gaspar, 500 - 30535-610 - Belo Horizonte - MG - Brazil
email: marcello@pucminas.br

Summary. *Recent experimental studies on the nonlinear evolution of three-dimensional wavetrains have shown the development of a mean flow distortion that does not decay despite the decay of the fundamental disturbances after the second branch of the instability loop. This mean flow distortion has a spanwise structure consisting of positive and negative regions distributed like longitudinal streaks, which become more complex as the nonlinearity develops. In order to gain a better insight into the flow physics of this problem numerical simulations are also being carried out. The numerical model is based on the Parabolized Stability Equations* (PSE) *and takes into account both nonlinear and non-parallel effects. The numerical computations show that the nonlinear evolution of three-dimensional Tollmien-Schlichting waves results in the development of a mean flow distortion containing longitudinal vortices that decay very slowly after the decay of the initial disturbances. The strongest vortices have a spanwise wavenumber two times the wavenumber of the fundamental Tollmien-Schlichting waves. The resulting structure resembles the structure observed experimentally and the splitting of longitudinal streaks observed at latter stages is captured by the computation. The numerical results provide additional data on which to build a theoretical model of the mechanisms involved.*

Keyword: *Boundary layer instability, Laminar-turbulent transition, Wavetrains, Parabolized stability equations, Tollmien-Schlichting waves.*

1. INTRODUCTION

The natural transition from laminar to turbulent flow in shear layers observed in aeronautical applications usually takes place in an environment containing random noise.

The transition process leading to turbulence in such applications may, therefore, be very distinct from the well established model that consists in the amplification of plane regular disturbances seeded by low amplitude oblique modes that interact nonlinearly and lead to secondary instability processes. Some studies show that indeed this is the case.

In an effort to try to understand the mechanisms involved in natural transition, recent investigations have been conducted (Gaster, 1978; Shaikh, 1997; Medeiros and Gaster, 1999a; Medeiros and Gaster, 1999b; Medeiros, 1998a; Medeiros, 1998b; Medeiros, 1997a; Medeiros, 1997b). These investigations focus on the nonlinear evolution of three-dimensional disturbances generated by a point source. This type of disturbances generate a continuous range of oblique modes, representing a more generic situation, closer to the three-dimensional conditions observed in actual engineering applications.

The results from these investigations have shown that the nonlinear evolution of three-dimensional wavetrains is not entirely consistent with the classic nonlinear theory developed for plane wavetrains. The first signature of nonlinearity is the formation of longitudinal streaks at relatively low wave amplitudes. Despite the decay of the fundamental disturbances after the waves have crossed the upper branch of the stability diagram, the longitudinal streaks do not decay and develop a more complex structure.

The current investigation presents numerical results aimed at understanding the nature of the longitudinal streaks resulting from the evolution of three-dimensional wavetrains. The numerical model is based on the Parabolized Stability Equations (PSE), which have been extensively used to simulate the nonlinear evolution of longitudinal vortical structures (Li and Malik, 1995; Mendonça *et al.* , 1998; Mendonça *et al.* , 1999) similar to the longitudinal streaks observed by Medeiros (1998a; 1998b; 1997a; 1997b).

2. FORMULATION

The Navier-Stokes equations for an incompressible flow of a Newtonian fluid are simplified by assuming that the dependent variables are decomposed into a mean component and a fluctuating component as follows:

$$\vec{u^*} = \vec{U^*} + \vec{u'^*}, \qquad \text{and} \qquad p^* = P^* + p'^*, \tag{1}$$

where $\vec{u^*} = [u^*, v^*, w^*]^T$ is the velocity vector and p^* is the pressure. The superscript '$*$' indicates dimensional variables.

The coordinate system used in the present work is based on the streamlines (ψ^*) and potential lines (ϕ^*) of the inviscid flow over a flat plate. The equations are nondimensionalized using δ_0^* and U_∞^* as the length and velocity scaling parameters, where $\delta_0^* = (\nu^*\phi_0^*/U_\infty^*)^{1/2}$ is the boundary layer thickness parameter, U_∞^* is the free stream velocity, ϕ_0^* is a reference length taken as the streamwise location where initial conditions are applied, and ν^* is the kinematic viscosity. The Reynolds number is defined as: $Re = U_\infty^*\delta_0^*/\nu^*$.

The mean flow is governed by Prandtl boundary layer equations for the flow over a flat plate. The resulting governing equations for the perturbations are elliptic and the perturbations propagate in the flow field as wave structures. The governing equations can be simplified if the wave like nature of the perturbations are represented by their frequency, wavenumber, and growth rate. The perturbation Φ' is assumed to be composed of a slowly varying shape function and an exponential oscillatory wave term. It is represented mathematically as a Fourier expansion truncated to a finite number of modes:

$$\Phi' = \sum_{n=-N}^{N} \sum_{m=-M}^{M} \Phi_{n,m}(\phi,\psi)\chi_{n,m}(\phi,z,t), \tag{2}$$

$$\chi_{n,m}(\phi,z,t) = \exp\left[\int_{\phi_0}^{\phi} a_{n,m}(\xi)d\xi + im\beta z - in\omega t\right], \tag{3}$$

$$a_{n,m}(\phi) = \gamma_{n,m}(\phi) + in\alpha(\phi). \tag{4}$$

where $\Phi_{n,m}(\phi,\psi) = [u_{n,m}, v_{n,m}, w_{n,m}, p_{n,m}]^T$ is the complex shape function vector. This procedure is similar to a normal mode analysis, but, in this case, the shape function $\Phi_{n,m}$ is a function of both ϕ and ψ.

The streamwise growth rate $\gamma_{n,m}$, the streamwise wavenumber α, and the spanwise wavenumber β were nondimensionalized using the boundary layer thickness parameter δ_0^*. The frequency ω was nondimensionalized using the free stream velocity U_∞^* and the boundary layer thickness parameter δ_0^* .

The perturbation variable Φ', as defined in Eq. (2), is substituted in the governing equations which are then simplified by assuming that the shape function, wavelength, and growth rate vary slowly in the streamwise direction. Second order derivatives and products of first order derivatives can, therefore, be neglected. After performing a harmonic balance in the frequency, a set of coupled nonlinear equations is obtained. These resulting equations are known as the Parabolized Stability Equations (PSE). For each mode (n,m) the equation in vector form results:

$$\overline{A}_{n,m}\Phi_{n,m} + \overline{B}_{n,m}\frac{\partial \Phi_{n,m}}{\partial \phi} + \overline{C}_{n,m}\frac{\partial \Phi_{n,m}}{\partial \psi} + \overline{D}_{n,m}\frac{\partial^2 \Phi_{n,m}}{\partial \psi^2} = \frac{\overline{E}_{n,m}}{e^{\int_{\phi_0}^{\phi} a_{n,m}(\xi)d\xi}}, \tag{5}$$

where the coefficient matrices can be found in Mendonça (1997).

The resulting equations are parabolic in ϕ and the solution can be marched downstream given initial conditions at a starting position ϕ_0. The approach is correct as long as the instabilities are convective and propagate in the direction of the mean flow not affecting the flow field upstream.

The boundary conditions for Eq. (5) are given by homogeneous Dirichlet no-slip conditions at the wall, Neumann boundary conditions for the velocity components in the far field, and homogeneous Dirichlet condition for pressure in the far field.

For the parabolic formulation, it is necessary to specify initial conditions at a starting position ϕ_0 downstream of the stagnation point at the leading edge of the curved plate. The initial conditions are obtained from Orr-Sommerfeld solutions.

2.1 Normalization condition

The splitting of the perturbation $\Phi'(\phi,\psi,z,t)$ in Eq. (2) into two functions, $\Phi_{n,m}(\phi,\psi)$ and $\chi_{n,m}(\phi,\psi,z,t)$, is ambiguous, since both are functions of the streamwise coordinate ϕ. It is necessary to define how much variation will be represented by the shape function $\Phi_{n,m}(\phi,\psi)$, and how much will be represented by the exponential function $\chi_{n,m}(\phi,\psi,z,t)$. This definition has to guarantee that rapid changes in the streamwise direction are avoided so that the hypothesis of slowly changing variables is not violated. The objective is to transfer fast variations of $\Phi_{n,m}(\phi,\psi)$ in the streamwise direction to the streamwise complex wavenumber $a_{n,m}(\phi) = \gamma_{n,m}(\phi) + in\alpha(\phi)$. If this variation is represented by $b_{n,m}$, for each step in the streamwise direction it is necessary to iterate on $a_{n,m}(\phi)$ until $b_{n,m}$ is smaller than a given threshold. At each iteration k, $a_{n,m}(\phi)$ is updated according to:

$$(a_{n,m})_{k+1} = (a_{n,m})_k + (b_{n,m})_k. \tag{6}$$

The variation $b_{n,m}$ of the shape function can be monitored in different ways. In the present implementation the following is used:

$$b_{n,m} = \frac{1}{\int_0^\infty \|\vec{u}_{n,m}\|^2 d\psi} \int_0^\infty \left(\vec{u}^{\dagger}_{n,m} \cdot \frac{\partial \vec{u}_{n,m}}{\partial \phi} \right) d\psi, \tag{7}$$

where $\vec{u}^{\dagger}_{n,m}$ is the complex conjugate of $\vec{u}_{n,m}$. The integral of $\|\vec{u}_{n,m}\|^2$ was used to assure that the variation is independent from the magnitude of $\vec{u}_{n,m}$.

2.2 Numerical method

The system of parabolic nonlinear coupled equations given by Eq. (5) is solved numerically using finite differences. The partial differential equation is discretized implicitly using a second order backward differencing in the streamwise direction, and fourth order central differencing in the normal direction. The resulting coupled algebraic equations form a block pentadiagonal system which is solved by LU decomposition.

To start the computation a first order backward differencing is used. The first order approximation is used also in a few subsequent steps downstream in order to damp numerical transients more efficiently. For the points neighboring the boundaries, second order central differencing in the normal direction was used.

The nonlinear terms are evaluated iteratively at each step in the streamwise direction. The iterative process is used to enforce both the normalization condition and the convergence of the nonlinear terms. A Gauss-Siedel iteration with successive overrelaxation is used. The nonlinear products are evaluated in the time domain. The dependent variables in the frequency domain are converted to the time domain by an inverse Fast Fourier Transform subroutine. The nonlinear products are evaluated and the results are transformed back to the frequency domain.

The complex wavenumber is updated at each iteration according to Eq. (6), and the variation in the shape function is monitored through Eq. (7). The iteration is considered converged when the normalization condition is no larger than a given small threshold. In the present implementation this threshold is 10^{-8}.

Results from the present numerical implementation of the PSE have been compared to experimental and numerical results for K-type breakdown, H-type breakdown and for the nonlinear development of Görtler Vortices. The code was able to reproduce the nonlinear development of interacting disturbances with good accuracy.

3. RESULTS AND DISCUSSION

Some results of the experimental investigation on the evolution of disturbances emanating from a point source conducted by Medeiros (1998a; 1998b; 1997a; 1997b) are show in Fig. 1 and Fig. 2. The initial amplitude of the disturbances is such that they grow and decay consistent with the linear stability analysis. Measurements along the centerline of the plate reveal that a mean flow distortion (MFD) evolves and does not decay after the decay of the Tollmien-Schlichting (TS) modes. Figure 1 shows that this MFD is initially negative, but switches to positive further downstream. In order to gain more insights on the evolution of the MFD additional measurements were taken along the spanwise direction. Those measurements revealed the development of longitudinal streaks composed of alternating positive and negative MFD. It was observed that, as the disturbances travel downstream a positive streak appears at the centerline in what seems to be a splitting of a given negative streak as seen in Fig. 2.

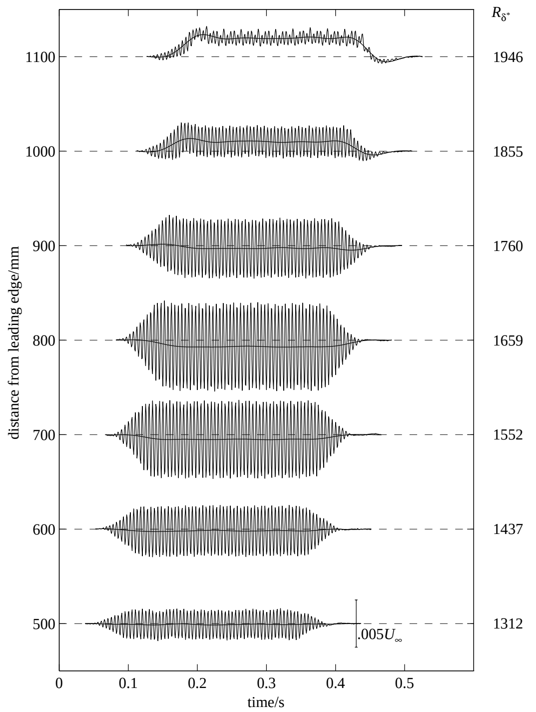

Figure 1: evolution of three-dimensional wavetrains along the centerline of a plate.

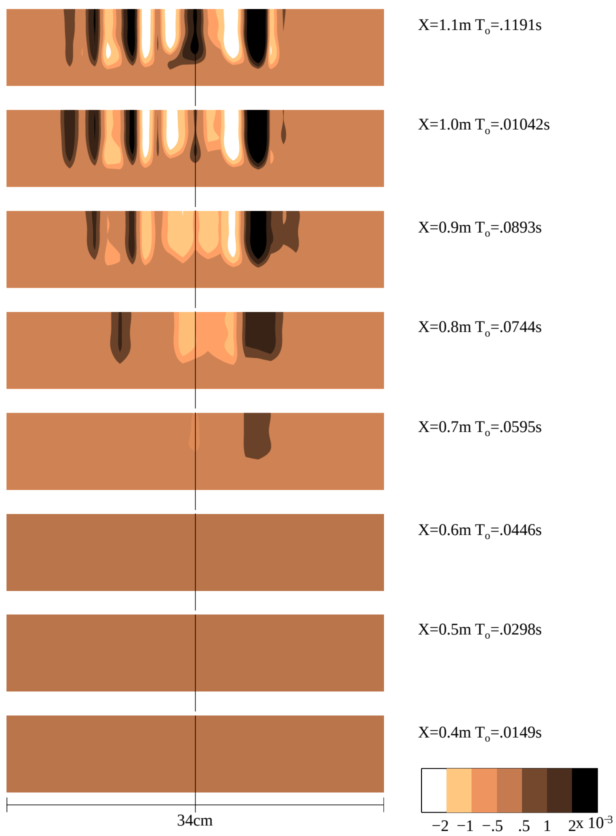

Figure 2: Streaks generated by three-dimensional wavetrains.

The Fourier expansion in the PSE formulation considers a few fundamental modes and higher harmonics or subharmonics of lower amplitudes. In this way, the development of an idealized wavetrain generated by a point source has to be modeled by a finite number of oblique waves and a planar wave, all with the same initial amplitude. At first, nine oblique modes were tested, but it was not possible to reach convergence in the computation. Therefore, in order to comply more closely with the PSE Fourier expansion, the point source was modeled by five fundamental waves, namely, a two-dimensional wave given by mode (1,0), a pair of oblique waves given by modes (1,1) and (1,-1) with the same frequency of the 2D wave, and a second pair of oblique waves given by modes (1,2) and (1,-2), with the same frequency of the other two waves, but with a spanwise wavenumber twice the wave number of the oblique wave (1,1). The rationale behind this model is that the higher the spanwise wavenumber the more stable is the oblique wave and soon after the first steps in the downstream direction these waves would decay rapidly, contributing very little to the nonlinear evolution.

The parameters used in the computation are taken from Medeiros (1997a). The free stream velocity is $U_0 = 16.7$ m/s, the frequency is $\nu = 200$ Hz. The initial conditions are imposed upstream of the lower branch of the stability diagram at $x_0 = 0.2149$ m. At this position the boundary layer thickness parameter is $\delta = 4.29 \times 10^{-4}$ m and the corresponding Reynolds number is $Re = 500$. The resulting nondimensional frequency is $\omega = 2\pi\nu\delta/U_0 = 0.03235$ ($F = 10^6\omega/Re = 64.69$). The spanwise wavenumbers are $\beta_{1,1} = 0.05$ ($b_{1,1} = 10^{-3}\beta/Re = 0.1$), and $\beta_{1,2} = 0.1$ ($b_{1,2} = 10^{-3}\beta/Re = 0.2$). The disturbance initial amplitude is $\epsilon = .01\%$ of the free stream velocity.

Note that in the PSE formulation the length scale is the boundary layer thickness parameter. That differs from the displacement thickness (the length scale used by Medeiros) by a factor of 1.7204.

Figure 3 shows the variation of the amplitude of the streamwise velocity component along the streamwise direction. The amplitude is measured at a constant nondimensional distance from the wall $\psi = 1$. It shows that the nonlinear evolution does result in the growth of a MFD which can be decomposed into a true mean flow distortion given by Fourier mode (0,0) and by a spanwise periodic mean flow distortion given by modes (0,m), $m \neq 0$. The (0,m) modes are characteristic of counter rotating longitudinal vortices and the strongest mode has a spanwise wavelength two times the wavelength of the fundamental mode (1,1). The simulation indicates that, even after the decay of the fundamental TS waves the MFD does not decay. This result is in agreement with the experimental observation, but this particular numerical test case does not show the change in the MFD from negative to positive.

Another test case was set up motivated by the fast decay of mode (1,2) as seen in Fig. 3. Since the amplitude of mode (1,2) starts to decay much earlier than the amplitude of the other fundamental modes, it was thought that a simpler model based on the interaction of mode (1,0), (1,1) and (1,-1) could represent the physics of the nonlinear process just as well. Figure 4 shows the evolution of the amplitude of different Fourier modes. It can be observed that the absence of mode (1,2) in the computation does not affect the development of mode (1,0) and (1,1). The growth of mode (0,2) is more regular, while mode (0,0) is very little affected. Mode (0,4) grows more slowly upstream but reaches the same amplitude at the streamwise position $Re = 1250$. Again, this test case results in the the growth of a MFD that does not decay after the decay of the fundamental TS waves.

The curve labeled "(1,2) a" in Fig. 4 indicates the growth of mode (1,2) from the first test case, while the curve labeled "(1,2)" shows the growth of mode (1,2) do to the nonlinear interaction between mode (1,0) and (1,1) in the second test case. It is interesting to observe that the development of mode (1,2) in the first test case is attracted by the forcing generated by the interaction of modes (1,0) and (1,1). This attraction by different nonlinear interaction forcing mechanisms explain the abrupt change in the growth rate observed in different Fourier modes and, possibly the irregular growth of mode (0,2). In other words, the development of a given Fourier mode is governed at different stages of the development by different forcing mechanisms, resulting in different growth rates.

Additional computations revealed that the nonlinear evolution of a single pair of oblique waves given by modes (1,1) and (1,-1) also results in the development of a MFD. In this case, the nonlinear evolution will have less forcing mechanisms do to the presence of less fundamental modes. This is confirmed by more smooth amplitude curves as seen in Fig. 5. Certain modes do not develop, like mode (1,2), (0,1), (0,3) and the strongest longitudinal mode is again mode (0,2). Despite the fact that this model is much simpler than the experimental point source the result again shows the development of a MFD

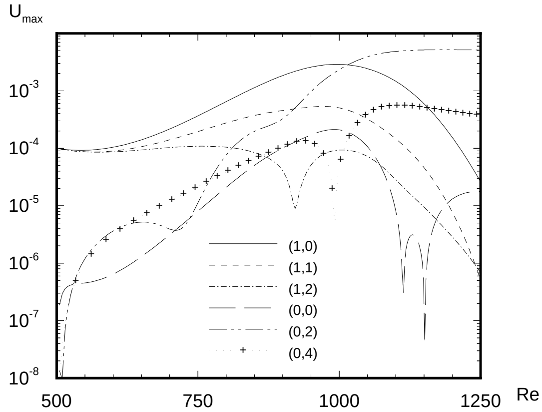

Figure 3: Maximum amplitude of different Fourier modes. Initial disturbance composed of modes (1,0), (1,1), (1,-1), (1,2) and (1,-2)

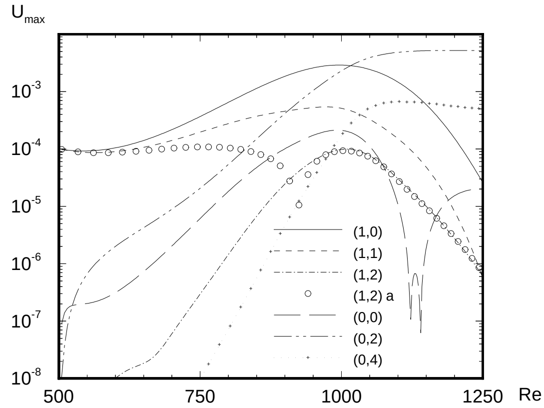

Figure 4: Maximum amplitude of different Fourier modes. Initial disturbance composed of modes (1,0), (1,1) and (1,-1)

that does not decay after the decay of the fundamental mode (1,1). It indicates that the fundamental mechanism for the formation of the longitudinal streaks is related to the development of three-dimensional disturbances, but not necessarily to the nonlinear interaction of different oblique modes.

All the results obtained so far show the development of a MFD composed of a true MFD and spanwise periodic longitudinal vortices. These are related to the longitudinal streaks observed experimentally but did not result in a change in the disturbance from positive to negative as observed experimentally. Iso-velocity contours for the MFD in the spanwise plane are presented in Fig. 6. It shows the spanwise distribution of the streamwise disturbance velocity. This spanwise structure is preserved at different streamwise positions, not showing any type of vortex splitting or vortex doubling.

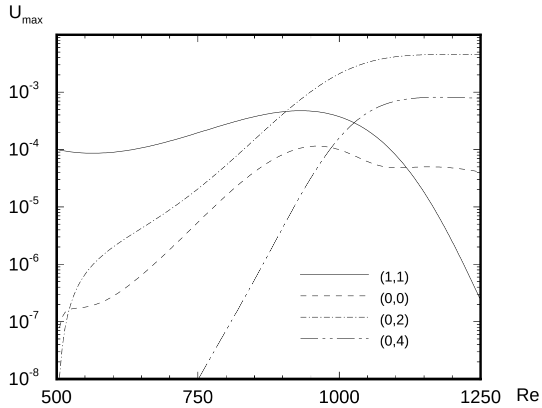

Figure 5: Maximum amplitude of different Fourier modes. Initial disturbance composed of modes (1,1) and (1,-1)

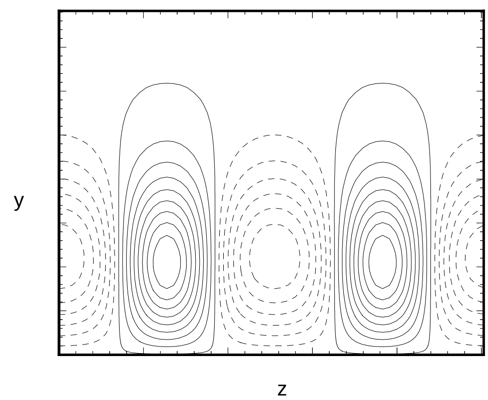

Figure 6: Mean flow distortion velocity contours in the spanwise plane. Initial amplitude of $\epsilon = .01\%$

The change of sign in the MFD may be related to the development of vortices higher harmonics. In order to test this hypothesis the initial amplitude of the fundamental oblique waves was increased to $\epsilon = .025$, increasing the nonlinear interaction. Figure 7 shows that the results for the development of the oblique waves with higher initial amplitude results in a modification of the spanwise structure downstream do to the stronger nonlinear interaction. The negative velocity streak seems to split into two. Although it was not possible to see the formation of a positive streak between the two new streaks, the resulting shape indicates that the negative streak is being pushed from below.

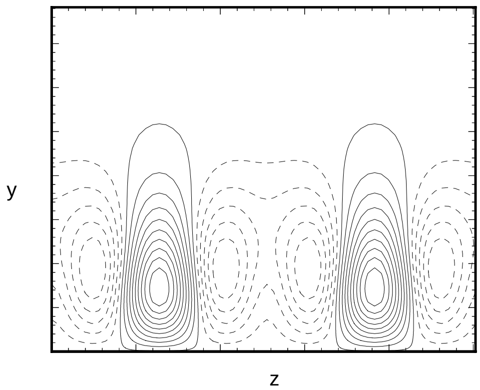

Figure 7: Mean flow distortion velocity contours in the spanwise plane. Initial amplitude of $\epsilon = .025\%$

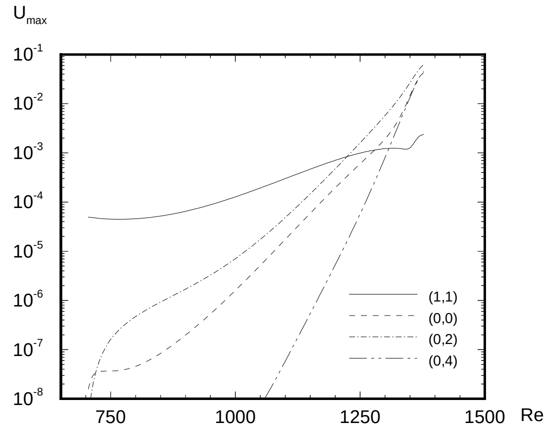

Figure 8: Maximum amplitude of different Fourier modes. $\omega = 0.0245$.

The previous test cases show that the forcing mechanism for the development of the MFD is the nonlinear interaction of a pair of oblique waves. It would be interesting to have the forcing mechanism active along a longer distance in the streamwise direction. To do so, it is necessary to have a pair of waves that travel a longer distance in the streamwise direction before crossing the upper branch of the neutral curve. The idea is to have fundamental oblique waves of lower initial amplitudes resulting in weaker nonlinear interaction but forcing the MFD during a longer streamwise distance.

The flow conditions for the previous test cases were changed by reducing the frequency such that the disturbances travel a longer distance in the unstable region of the instability diagram. For the following test cases the frequency was reduced to $\omega = 0.0245$. The Reynolds number was increased to $Re = 700$ in order to keep the initial conditions close to the lower branch of the neutral curve.

Using the above initial conditions a number of test cases were run. Starting with an initial amplitude of $\epsilon_{1,1} = 0.04$ a very strong nonlinear interaction was observed that resulted in computational convergence problems. The evolution of longitudinal vortices was observed, but do to the nonlinearity strength it was not possible to detect any kind of vortex doubling or vortex splitting before the computation failed to converge.

In order to comply more closely with the experimental results, where the fundamental three-dimensional waves grow and decay according to linear theory, the initial amplitude was reduced to $\epsilon_{1,1} = 0.005$. Figure 8 show the amplitude evolution of the fundamental mode (1,1) and modes (0,0), (0,2), and (0,4). It shows that the modes related to the MFD grow to amplitudes one order of magnitude higher than the fundamental mode. Figures 9 and 10 show the velocity contours in the spanwise plane at the streamwise positions given by $Re = 1200$ and $Re = 1377$ respectively. It can be observed that the spanwise structure changes considerably where a positive streak seems to grow from the

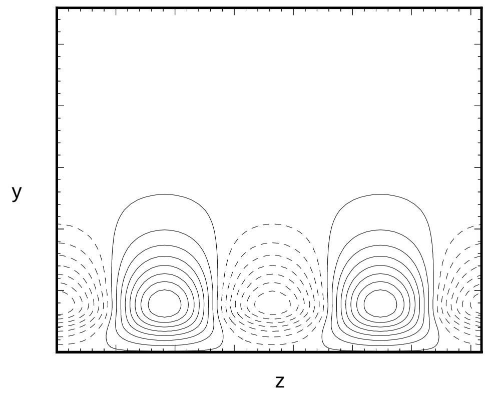

Figure 9: Mean flow distortion velocity contours in the spanwise plane.
$Re = 1200$. $\omega = 0.0245$.

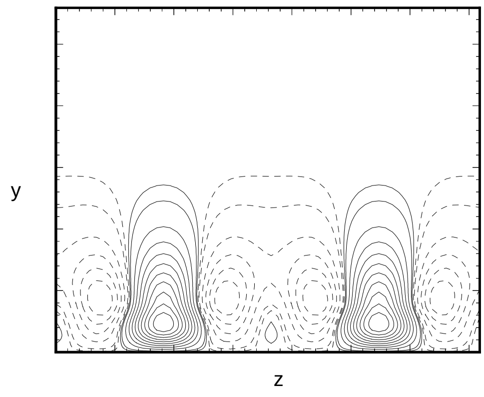

Figure 10: Mean flow distortion velocity contours in the spanwise plane.
$Re = 1377$. $\omega = 0.0245$.

wall upwards splitting the negative streak. This result is consistent with the change of sign of the MFD observed in the experimental results.

The following important conclusions can be drawn from the present results: a) It seems that the fundamental mechanism for the development and splitting of longitudinal streaks is related to the growth of oblique waves, but not necessarily to the interaction of different oblique modes generated by a point source. That allows the used of simple models to simulate the process. b) The longitudinal streaks have a wavelength twice the wavelength of the fundamental oblique mode. The splitting of negative streaks seems to be related to the development of higher harmonics in the nonlinear process and take place when a positive streak grows from the wall.

4. CONCLUSIONS

Using the Parabolized Stability Equations to model the evolution of wavetrains observed experimentally it was possible verify that a mean flow distortion does evolve and does not decay after the decay of the fundamental disturbances. The variation of the spanwise structure along the streamwise direction observed experimentally was also captured by the numerical results. The results suggest that a positive streak grows from the wall, splitting a negative streak into two.

If the initial amplitude of the fundamental disturbances is high the nonlinearity is too strong and breakdown to turbulence takes place before the growth of the mean flow distortion. On the other hand, if the nonlinearity is too weak the mean flow distortion does not grow enough to result in vortex splitting or vortex doubling.

The results show that the evolution of the mean flow distortion depends on the evolution of three-dimensional structures, but not necessarelly on structures generated by a point source with its continuous spectrum of oblique modes. Using a pair of oblique waves it was possible to capture the basic features of the phenomena observed experimentally.

Additional investigations are underway to find the theoretical bases for this nonlinear behavior.

5. ACKNOWLEDGMENT

The first author would like to acknowledge the financial support received from FAPESP (Fundação de Amparo à Pesquisa dos Estado de São Paulo) for the financial support received for the participation in the XVII COBEM.

REFERENCES

Gaster, M. 1978. The physical process causing breakdown to turbulence. *In: 12th Naval Hydrodynamics Symposium, Washington.*

Li, F., and Malik, M. R. 1995. Fundamental and Subharmonic Secondary Instabilities of Görtler Vortices. *J. Fluid Mechanics*, **297**, 77–100.

Medeiros, M. A. F. 1997a. Laminar-turbulent transition: the nonlinear evolution of three-dimensional wavetrains in a laminar boundary layer. *In: Proceedings of the XIV COBEM, Brazilian Congress on Mechanical Engineering.*

Medeiros, M. A. F. 1997b. Nonlinear mean flow distortion caused by a wavetrain emanating from a harmonic point source in a flat plate boundary layer. *Pages 34a–34b of: EUROMECH Colloquium 359, Stability and Transition of Boundary Layer Flows.*

Medeiros, M. A. F. 1998a. Nonlinear evolution of a three-dimensional wave train in a flat plate boundary layer. *In: 21st ICAS, Congress of the International Council of Aeronautical Sciences.*

Medeiros, M. A. F. 1998b. Transition to turbulence of low amplitude three-dimensional disturbances in flat plate boundary layers. *Pages 1346–1351 of: Proceedings of the VII ENCIT, Brazilian Meeting on Thermal Sciences.*

Medeiros, M. A. F., and Gaster, M. 1999a. The influence of phase on the nonlinear evolution of wavepackets in boundary layers. *The J. Fluid Mechanics.* Accepted for publication.

Medeiros, M. A. F., and Gaster, M. 1999b. The production of sub-harmonic waves in the nonlinear evolution of wavepackets in boundary layers. *The J. Fluid Mechanics.* Accepted for publication.

Mendonça, M. T. 1997. *Numerical Analysis of Görtler Vortices /Tollmien-Schlichting Waves Interaction With a Spatial Nonparallel Model.* Ph.D. thesis, The Pennsylvania State University.

Mendonça, M. T., Morris, P. J., and Pauley, L. L. 1998. Interaction Between Görtler Vortices and two-dimensional Tollmien-Schlichting waves. *Submited to The Physics of Fluids.*

Mendonça, M. T., Pauley, L. L., and Morris, P. J. 1999. Effect of wave frequency on Görtler Vortices Tollmien-Schlichting waves interaction. *In: 37th AIAA Aerospace Sciences Meeting and Exhibit.*

Shaikh, F. N. 1997. Investigation of transition to turbulence using white noise excitation and local analysis techniques. *J. Fluid Mechanics*, **348**, 29–83.